Lecture 1 (Proposition and its equivalence)

Proposition: A proposition is a declarative sentence which is either true or false but not both. Propositions are generally expressed by small alphabets $p, q, r, \ldots$.

Examples: 1: Paris is in France (true),

2: London is in Denmark (false),

3: $2 < 4$ (true),

4: $4 = 7$ (false).

However the following are not propositions:

1: what is your name? (this is a question),

2: do your homework (this is a command),

3: this sentence is false (neither true nor false),

4: x is an even number (it depends on what x represents),

5: Socrates (it is not even a sentence).

The truth or falsehood of a proposition is called its truth value

Compound Proposition: A proposition that is constructed by combining one or more propositions is called a compound proposition. We denote compound propositions by capital alphabets $L, M, X, Y, \ldots$. The propositions in a compound proposition are called primitives.

1. P: If you work hard, then you will get A grade. Here primitives are: $p :=$ You work hard, and $q :=$ You will get A grade.
2. Q: Amit is good in study and he plays football every day. Here $p :=$ Amit is good in study, $q :=$ Amit plays football everyday.

Connective: Connectives are used for making compound propositions. The main ones are the following (p and q represent given propositions):

Name	**Notation**	**Meaning**
Negation	$\neg\, p$	not p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	p or q
Exclusive OR	$p \oplus q$	either p or q, but not both
Implication	$p \Rightarrow q$	p implies q
Bi-conditional	$p \Leftrightarrow q$	p if and only if q

Truth Table: A table showing ouput (truth or falsity) of a proposition from all possible inputs (all combinations of Truth and False for the inputs). Let p, q be propositions.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Conditional statement: Let p and q be two propositions. The conditional proposition $p \Rightarrow q$ is the proposition "if p, then q". A conditional statement has two parts, one is hypothesis (p) and other is conclusion (q).

Example: If you do your homework, you will not be punished. Here, the hypothesis $p :=$ "you do your homework" and the conclusion $q :=$ "you will not be punished".

Inverse, Converse, Contra-positive: We can form new conditional propositions from an existing conditional proposition. These are: Inverse, Converse and Contra-positive. So if $p \Rightarrow q$ is a conditional proposition, then inverse is $\neg p \Rightarrow \neg q$, converse is $q \rightarrow p$, and contra-positive is $\neg q \Rightarrow \neg p$.

Tautology, Contradiction, Contingency: A compound statement which is always true is called a tautology. A compound statement which is always false, is called a contradiction. If a compound statement is neither tautology nor contradiction, then it is called contingency.

Example: Let p and q be propositions. Then $(p \wedge \neg p)$, $(p \vee \neg p)$ and $(p \wedge q)$ are contradiction, tautology and contingency respetively. To see this, construct their truth tables.

Example: Construct the truth table for compund proposition $(p \vee \neg q) \Rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \Rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Propositional Equivalence: Two propositions X and Y are logically equivalent or equivalent, denoted as $X \equiv Y$, if the bi-conditional proposition $X \Leftrightarrow Y$ is a tautology or if the columns giving their truth values agree.

Example: Show that $\neg(p \vee q) \equiv [(\neg p) \wedge (\neg q)]$.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

In the above truth table, we see that truth value of $\neg(p \vee q)$ and $[(\neg p) \wedge (\neg q)]$ are same (see columns six and seven) or $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ is a tautology. Therefore the propositions are equivalent.

Exercise: Show that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Laws of propositions: Let p, q, r be primitive statements.

1. **Double negation:** $\neg\neg p \equiv p$.
2. **De Morgan's Laws:** $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$
3. **Commutative Laws:** $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$.
4. **Associative Laws:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ and $p \vee (q \vee r) \equiv (p \vee q) \vee r$.
5. **Distributive Laws:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
6. **Idempotent Laws:** $p \wedge p \equiv p$ and $p \vee p \equiv p$
7. **Identity Laws:** $p \wedge T \equiv p$ and $p \vee F \equiv p$.
8. **Inverse Laws:** $p \wedge \neg p \equiv F$ and $p \vee \neg p \equiv T$
9. **Dominations Laws:** $p \vee T \equiv T$ and $p \wedge F \equiv F$,
10. **Absorption Laws:** $p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$

One can also show the equivalence of propositions by using the laws of propositions. Here are examples.

Example: Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

Solution: $\neg(p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg(\neg p \wedge q)$ (by De Morgan's Law)

$\equiv \neg p \wedge (p \vee \neg q)$ (by De Morgan's Law and Double negation Law)

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ (by Distributive Law)

$\equiv F \vee (\neg p \wedge \neg q)$ (by Inverse Law)

$(\neg p \wedge \neg q)$. (by Identity Law)

Exercise: Show that $p \Rightarrow q \equiv \neg p \vee q$.

Example: Show that $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology.

Solution: By above exercise: $(p \wedge q) \Rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ (by De Morgan's Law)

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ (by Associative and Commutative Laws)

$\equiv T \vee T$ ‘ (by Inverse Law)

$\equiv T$. (by Dominations Law)

Thus $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology.

Argument and its validity: An argument is a sequence of statements in which all the statements $p_1, p_2, \ldots, p_n$, except the final one, are called premises (or assumptions or hypothesis) and the final statement q is called the conclusion.

Validity of an argument: An argument is valid if the truth of all its premises implies that the conclusion is true or $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is a tautology. Here p_i's are premises or hypothesis and q is conclusion.

Example (formulation of argument):

If I read my text, I will understand how to do my homework.

I understand how to do my homework

Therefore I read my text.

Solution: First premise: If I read my text, I will understand how to do my homework.

Second premise: I understand how to do my homework.

Conclusion: I read my text.

Let us write its primitives:

t : I read my text.

u : I understand how to do my homework.

Thus symbolic form of an argument:

$t \to u$
u
$\therefore t.$

Now, we check the validity of the argument.

Method I

t	u	$t \Rightarrow u$ (premise 1)	u (premise 2)	t (conclusion)
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

A row, in which all the premises are true is called critical row. Here 1st and 3rd rows are critical. In the first row, conclusion is true while in the 3rd row it not true. So, the argument is not valid or invalid.

Method II Check that $[(t \to u) \wedge u] \to t$ is not a tautology.

Remark: Note that we can use truth table to determine whether an argument is valid or not. But this becomes tedious if it has large number of primitives. For example if an argument has 10 different primitives then it has 2^{10} rows. So, to avoid this, we first establish the validity of some relatively simple arguments. These arguments are called rule of inference, which can be used as building blocks to construct more complicated arguments.

Rules of inference

S. No.	Rule of Inference	Name
1.	$p \to q$ p $\therefore q$	Modus ponens
2.	$p \to q$ $\neg q$ $\therefore \neg p$	Modus tollens
3.	$p \to q$ $q \to r$ $\therefore p \to r$	Transitivity
4.	$p \vee q$ $p \vee q$ $\neg p$ OR $\neg q$ $\therefore q$ $\therefore p$	Elimination
5.	p OR q $\therefore p \vee q$ $\therefore p \vee q$	Addition
6.	$p \wedge q$ OR $p \wedge q$ $\therefore p$ $\therefore q$	Simplification
7.	p q $\therefore p \wedge q$	Conjunction
8.	$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

Example: Show that the premises:

It is not sunny this afternoon and it is colder than yesterday.
If we will go swimming then it is sunny.
If we do not go swimming, then we will take a canoe trip.
If we take canoe trip, then we will be home by sunset.

Lead to the conclusion

We will be home by sunset.

Solution: Let us first write the primitives:

p : It is sunny this afternoon.
q : It is colder than yesterday.
r : We will go swimming.
s : We will take a canoe trip.
t : We will be home by sunset.

Then premises becomes:

$\neg p \wedge q$
$r \to p$

$\neg r \to s$
$s \to t$

1: $\neg p \wedge q$ premise 1
2: $\neg p$ simplification using 1
3: $r \to p$ premise 2
4: $\neg r$ Mudus tollens using 2 and 3
5: $\neg r \to s$ premise 3
6: s Mudus ponens using 4 and 5
7: $s \to t$ premise 4
8: t Mudus ponens using 6 and 7

Thus conclusion is t.

Example: Show that the premises:

If you send me an e-mail, then I will finish writing the program.
If you do not send me an e-mail, then I will go to sleep early.
If I go to sleep early, then I will wake up feeling refreshed.

Lead to conclusion

If I do not finish writing the program, then I will wake up feeling refreshed.

Solution: First write its primitives:

p : You send me an email.
q : I will finish writing the program.
r : I will go to sleep early.
s : I will wake up feeling refreshed.

Premises:

$p \to q$
$\neg p \to r$
$r \to s$
$\neg q \to s$

1: $p \to q$ premise 1
2: $\neg q \to \neg p$ contrapositive of 1
3: $\neg p \to r$
premise 2
4: $\neg q \to r$ transitivity using 2 and 3
5: $r \to s$ ptemise 3
6: $\neg q \to s$ transitivity using 4 and 5.